Killara High School

Student Number: ____________________

2024 Higher School Certificate Trial

Mathematics Extension 2

General Instructions

- Reading time - 10 minutes
- Working time - 180 minutes
- Write using black pen
- Calculators approved by NESA may be used
- A reference sheet is provided
- For questions in Section II, show mathematical reasoning and/or calculations

Total Marks: 100

Section I – 10 marks (Pages 1–6)

- **Attempt Questions 1–10**
- **Allow about 15 minutes for this section**

Section II – 90 marks (Pages 7–13)

- **Attempt Questions 11–16**
- **Allow about 165 minutes for this section**

Section I

10 marks

Attempt Questions 1–10

Allow about 15 minutes for this section.

Use the multiple-choice answer sheet for Questions 1–10.

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1. From the statements given below, select the **TRUE** statement. **1**

   A. $y = \sin x \iff x = \sin^{-1} y$

   B. $A^2 = B^2 \implies A = B$

   C. $\exists\, x, y \in \mathbb{R} : \sqrt{x^2 + y^2} = x + y$

   D. $(A \cap B) \cup C = A \cap (B \cup C)$

2. The contrapositive of the the statement "All counters in this box are blue" is best given by: **1**

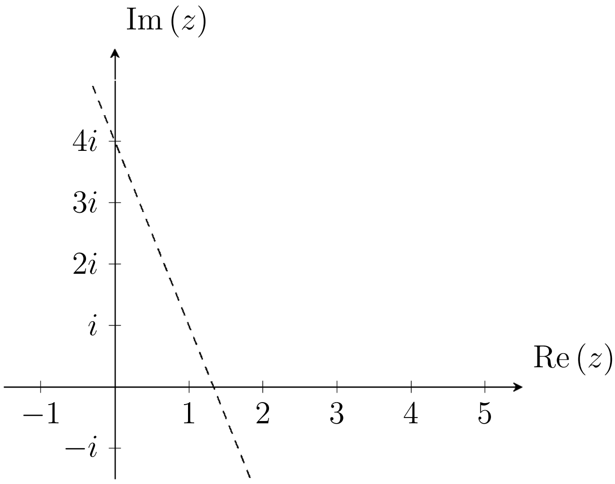
   A. No counters in this box are blue.

   B. All counters in this box are not blue.

   C. No counters that are not blue are in this box.

   D. All counters that are not blue are not in this box.

3. A shaded region on the complex plane is shown below. **1**



Which relation best describes the region shaded on the complex plane.

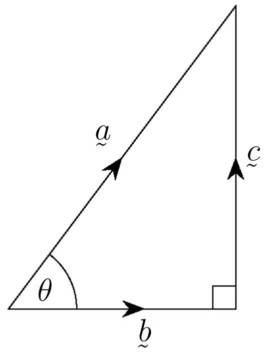
A. $|z+i| > |z-3|$

B. $|z+i| < |z-3|$

C. $|z-i| > |z+3|$

D. $|z-i| < |z+3|$

4. The right-angled triangle shown has sides represented by the vectors $\underset{\sim}{a}, \underset{\sim}{b}$ and $\underset{\sim}{c}$. **1**



Which of the following statements is **FALSE**?

A. $\underset{\sim}{b} \cdot (\underset{\sim}{a} - \underset{\sim}{c}) = |\underset{\sim}{b}|^2$

B. $\underset{\sim}{b} \cdot (\underset{\sim}{a} - \underset{\sim}{c}) = |\underset{\sim}{b}||\underset{\sim}{c}|$

C. $\underset{\sim}{a} \cdot \underset{\sim}{b} = |\underset{\sim}{a}||\underset{\sim}{b}| \cos(\theta)$

D. $\underset{\sim}{a} \cdot \underset{\sim}{c} = |\underset{\sim}{a}||\underset{\sim}{c}| \sin(\theta)$

5. Which of the following is equivalent to $\displaystyle\int x^5\sqrt{1-x^2}\,dx$? **1**

A. $\displaystyle\int \cos^5 x \sin x dx$

B. $\displaystyle\int \cos^5 x \sin^2 x dx$

C. $\displaystyle\int \sin^5 x - \sin^7 x dx$

D. $\displaystyle\int \sin^6 x - \sin^7 x dx$

6. A particle is moving in simple harmonic motion. A new force is applied that halves the period without changing the amplitude. What affect does this have on the magnitude of the velocity? **1**

   A. It remains unchanged.

   B. It halves.

   C. It doubles.

   D. It quadruples.

7. Which of the following is always true for non-zero complex nunmbers z_1, z_2? **1**

   A. $\text{Arg}\left(\dfrac{z_1}{z_2}\right) = \text{Arg}(z_1) - \text{Arg}(z_2)$, where $\text{Arg}\,(z)$ is the primary argument.

   B. $\text{Arg}(z_1 z_2) = \text{Arg}(z_1) + \text{Arg}(z_2)$ where $\text{Arg}(z)$ is the primary argument.

   C. $z_1 = r_1 e^{i\theta_1}, z_2 = r_2 e^{i\theta_2} \Rightarrow \dfrac{z_1}{z_2} = \dfrac{r_1}{r_2} e^{(\theta_1 - \theta_2 - 2\pi)i}$

   D. $z_1 = x_1 + iy_1, z_2 = x_2 + iy_2 \Rightarrow \text{Arg}(z_1 + z_2) = \tan^{-1}\left(\dfrac{y_1 + y_2}{x_1 + x_2}\right)$

8. A local politician spoke at the opening of a new school, saying, “If young people have access to good schools then they will become valued members of society!” **1**

Taking the converse and then contrapositive of this statement, you would get:

A. If young people do not have access to good schools then they will not become valued members of society.

B. If young people do not become valued members of society then they did not have access to good schools.

C. If young people become valued members of society then they had access to good schools.

D. If young people do not have access to good schools then they will become valued members of society.

9. Which of the following has the largest value? **1**

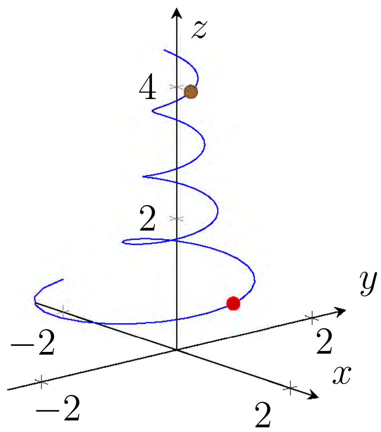
A. $\displaystyle\int_0^2 \left(x^2-4\right)\sin^8 x dx$

B. $\displaystyle\int_0^{2\pi} (2+\cos x)^3\, dx$

C. $\displaystyle\int_0^{2\pi} \sin^4 x dx$

D. $\displaystyle\int_0^{8\pi} 108\left(\sin^3 x-1\right) dx$

10. A curve follows a hyperbolic spiral such that $r = \dfrac{a}{\theta}$ in the xy plane and wraps around the z axis anticlockwise exactly three times for $z \in [1, 4]$. We are given that the point $P(1, 0, 1)$ lies on the curve. **1**



Which of the following best describes the curve?

A. $\left(\dfrac{\sin 2\pi t}{t}, \dfrac{\cos 2\pi t}{t}, t\right)$

B. $\left(\dfrac{\sin 4\pi t}{t}, \dfrac{\cos 4\pi t}{t}, t\right)$

C. $\left(\dfrac{\cos 4\pi t}{t}, \dfrac{\sin 4\pi t}{t}, t\right)$

D. $\left(\dfrac{\cos 4\pi t}{2t}, \dfrac{\sin 4\pi t}{2t}, 2t\right)$

Section II

90 marks
Attempt Questions 11–16
Allow about 165 minutes for this section.
Answer each question in the appropriate writing booklet.
All necessary working should be shown in every question.

Question 11 (15 Marks) Use a SEPARATE writing booklet.

(a) Express $\dfrac{1+i}{\sqrt{3}-i}$ in the form $a+ib$. **2**

(b) Find $\displaystyle\int \sec^7 x \tan x dx$. **2**

(c) Let $w = 2e^{i\frac{\pi}{3}}$.

(i) Write w^4 in the form $a+ib$, where $a, b \in \mathbb{R}$. **2**

(ii) Find the smallest integer $k > 4$ such that w^k is a real number. **1**

(d) Find the vector equation of the line through the point $A(6, -5, 1)$ perpendicular to, and intersecting, the vector equation $\underset{\sim}{a} = \lambda(-3, 2, -2)$. **2**

(e) By using the substitution $t = \tan\left(\dfrac{x}{2}\right)$ find: **3**

$$\int \frac{1}{\cos x - 2\sin x + 3} dx.$$

(f) Find the solutions to $z^2 - 8z + 25 = 0$ where z is a complex number. **1**

(g) A mass is attached to a spring. It is pulled down and then released, after which it begins oscillating in simple harmonic motion. Initially the mass has a height of $2\sqrt{5}$ cm above the ground before being released and it reaches its highest point $4\sqrt{5}$ cm after $\dfrac{\pi}{6}$ seconds. **2**

Find an equation for the height above the ground y in terms of t.

End of Question 11

Question 12 (16 Marks) Use a SEPARATE writing booklet.

(a) (i) If $a, b, c > 0$, prove that: **2**

$$a^2 + b^2 + c^2 \geqslant bc + ca + ab$$

(ii) Hence, or otherwise, prove that: **2**

$$2\left(a^3 + b^3 + c^3\right) \geqslant bc(b + c) + ca(c + a) + ab(a + b)$$

(b) Prove that $\sqrt{15}$ is irrational. **3**

(c) At times t, the position vectors of two points, P and Q, are given by: **3**

$$\underset{\sim}{p} = 2t\mathbf{i} + (3t^2 - 4t)\mathbf{j} + t^3\mathbf{k}$$

$$\underset{\sim}{q} = t^3\mathbf{i} - 2t\mathbf{j} + (2t^2 - 1)\mathbf{k}$$

Find the velocity and acceleration of Q relative to P when $t = 3$.

(d) Find $\displaystyle\int \ln\left(x^2 - 1\right) dx$. **3**

(e) Given that $3 - i$ is a root of the polynomial $P(x) = 3x^4 - 6x^3 - 27x^2 + 30x + 150$,

(i) Explain why $3 + i$ is also a root of the polynomial. **1**

(ii) Find all remaining roots of $P(x)$. **2**

End of Question 12

Question 13 (14 Marks) Use a SEPARATE writing booklet.

(a) Prove or refute the following: **2**

For any list of primes $p_1, \ldots, p_n$, the number $(p_1 p_2 \cdots p_n) + 1$ is prime.

(b) Prove by mathematical induction that $x^n - y^n$ is divisible by $x + y$ when n is even. **3**

(c) Let $\underset{\sim}{u} = \underset{\sim}{i} + \underset{\sim}{j} + z\underset{\sim}{k}$ and $\underset{\sim}{v} = 2\underset{\sim}{i} - \underset{\sim}{j} + 3\underset{\sim}{k}$. **3**

Find all z such that the angle between $\underset{\sim}{u}$ and $\underset{\sim}{v}$ is $\dfrac{\pi}{3}$.

(d) For z, w, complex numbers lying on the unit circle, prove that $\dfrac{z - w}{1 - zw}$ is real. **3**

(e) A particle is fired vertically upwards from the ground with an initial velocity $\underset{\sim}{u}$. It experiences a force from air resistance proportional to the square of it's velocity, $|F| = 0.098mv^2$, as well as the gravitational force. **3**

Show that:

$$y = \frac{250}{49} \log_e \frac{100 - u^2}{100 - v^2}.$$

You may assume $g = 9.8 \text{ ms}^{-2}$.

End of Question 13

Question 14 (16 Marks) Use a SEPARATE writing booklet.

(a) Prove that, for any integer greater than one, there is only one prime factorisation. **3**

(b) (i) Show by integrating both sides that **1**

$$\int_a^b f(x)dx = \int_a^b f(a+b-x)dx.$$

(ii) Hence, or otherwise, evaluate $\displaystyle\int_{-1}^{1} \frac{x^2}{1+e^x}dx$. **2**

(c) Find the shortest distance between the lines $\overrightarrow{r_1} = (\hat{i} + 2\hat{j} + \hat{k}) + \lambda(\hat{i} - \hat{j} + \hat{k})$ and $\overrightarrow{r_2} = (2\hat{i} - \hat{j} - \hat{k}) + \mu(2\hat{i} + \hat{j} + 2\hat{k})$. **3**

(d) If z_1 and z_2 are complex numbers such that $|z_1 - 5 + 3i| \leqslant 4$ and $|z_2 - 5i| \leqslant 2$, find the maximum and minimum values of $|z_1 - z_2|$. **3**

(e) Find $\displaystyle\int e^x\sqrt{10e^x - e^{2x}}dx$. **4**

End of Question 14

Question 15 (14 Marks) Use a SEPARATE writing booklet.

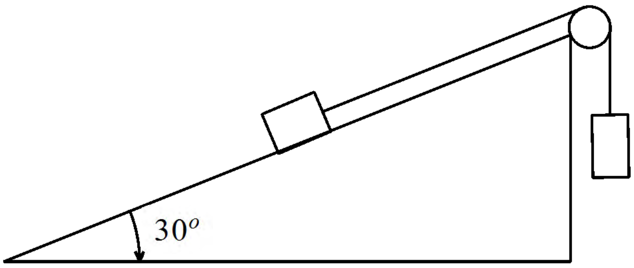
(a) Scientists use a pressure-sensitive device which measures depths as it sinks towards the seabed. The device of mass 2 kg is released from rest at the ocean's surface and as it sinks in a vertical line, the water exerts a resistance of $4v$ newtons to its motion, where $v\,\mathrm{m\,s^{-1}}$ is the velocity of the device t seconds after release.

(i) Draw a diagram showing the forces acting on the device and show that $a = g - 2v$. **2**

(ii) Find an expression for t in terms of g and v. **2**

(iii) State the terminal velocity. **1**

(b) Two masses of 5 kg and 2 kg are connected by a light inextensible string. The string is placed over a pulley, such that the 5 kg mass is resting on a rough plane inclined at 30° and the 2 kg mass is hanging under the pulley. The two masses are at rest before being released.



(i) Draw the forces acting on each mass. **2**

(ii) If the coefficient of friction is 0.3 find the net force on the 5 kg mass. **3**

(c) For $I_n = \displaystyle\int_0^a (a-x)^n \cos x\,dx, a > 0, n \geqslant 0,$

(i) Show that, for $n \geqslant 2$, **3**

$$I_n = na^{n-1} - n(n-1)I_{n-2}$$

(ii) Hence evaluate $\displaystyle\int_0^{\frac{\pi}{2}} \left(\frac{\pi}{2} - x\right)^2 \cos x\,dx$ **1**

End of Question 15

Question 16 (15 Marks) Use a SEPARATE writing booklet.

(a) Let the points $A_1, A_2, \ldots, A_n$ represent the nth roots of unity, $w_1, w_2, \ldots, w_n$, and suppose P represents any complex number z such that $|z| = 1$.

(i) Prove that $w_1 + w_2 + \cdots + w_n = 0$. **1**

(ii) Show that $|PA_i|^2 = (z - w_i)(\overline{z} - \overline{w_i})$ for $i = 1, 2, \ldots, n$. **1**

(iii) Prove that $\displaystyle\sum_{i=1}^{n} |PA_i|^2 = 2n$. **3**

(b) Let $f(x) = 1 + x^2$ and let x_1 be a real number.

For $n = 1, 2, 3, \ldots$, define

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

[You may assume that $f'(x_n) \neq 0$.]

(i) Show that **1**

$$|x_{n+1} - x_n| \geqslant 1 \text{ for } n = 1, 2, 3, \ldots.$$

(ii) Graph the function $y = \cot\theta$ for $0 < \theta < \pi$. **2**

(iii) Using your graph from part (ii), show that there exists a real number θ_n such that $x_n = \cot\theta_n$ where $0 < \theta_n < \pi$. **1**

(iv) Deduce that $\cot\theta_{n+1} = \cot 2\theta_n$ for $n = 1, 2, 3, \ldots$. **2**

$\left[\text{You may assume that } \tan 2\theta = \dfrac{2\tan\theta}{1 - \tan^2\theta}.\right]$

(v) Find all points x_i such that, for some n, $x_1 = x_{n+1}$. **4**

END OF PAPER

1. C
2. D
3. A
4. B
5. C
6. C
7. C
8. A
9. B
10. D

11 a) $\frac{1+i}{\sqrt{3}-i} = \frac{(1+i)(\sqrt{3}+i)}{4}$

$= \frac{\sqrt{3}-1+i+\sqrt{3}i}{4}$

$= \frac{\sqrt{3}-1}{4} + \frac{1+\sqrt{3}}{4}i$ — 1

b) $\int \sec^7 x \tan x \, dx$ let $u = \sec x$

$du = \sec^2 x \sin x \, dx$

$\int u^6 dx$ $du = \sec x \tan x \, dx$

$= \frac{u^7}{7} + C$

$= \frac{\sec^7 x}{7} + C$

[Could also have used $(1+\tan^2 x = \sec^2 x)$

c i) $w = 2e^{i\frac{\pi}{3}}$

$w^4 = \left(2e^{i\frac{\pi}{3}}\right)^4$

$= 16e^{i\frac{4\pi}{3}}$

$= 16\cos\frac{4\pi}{3} + 16i\sin\frac{4\pi}{3}$

$= -8 - 8\sqrt{3}i$

c ii) ~~For real~~

$w^k = 2^k e^{i\frac{k\pi}{3}}$

For real

$\frac{k\pi}{3} = n\pi$

$n \in \mathbb{Z}$

$k\pi = 3n\pi$

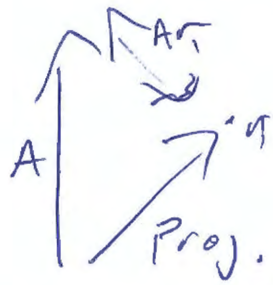
$k = 3, 6, 9, 12 \ldots$

$= 6 \ (k > 4)$

d) $\text{proj}_{\begin{pmatrix}-3\\2\\-2\end{pmatrix}} \begin{pmatrix}6\\-5\\1\end{pmatrix}$

$$= \frac{-3\times 6 + 2\times -5 + 2\times 1}{3^2+2^2+2^2} \begin{pmatrix}-3\\2\\-2\end{pmatrix}$$

$$= \frac{-30}{17} \begin{pmatrix}-3\\2\\-2\end{pmatrix}$$



$$A_r = \frac{-30}{17} \begin{pmatrix}-3\\2\\-2\end{pmatrix} - \begin{pmatrix}6\\5\\1\end{pmatrix}$$

$$= \begin{pmatrix}-\frac{12}{17}\\ \frac{25}{17}\\ \frac{43}{17}\end{pmatrix}$$

$$\vec{r} = \begin{pmatrix}6\\-5\\1\end{pmatrix} + \lambda \begin{pmatrix}-12\\25\\43\end{pmatrix}$$

c) $\displaystyle\int \frac{1}{\cos x - 2\sin x + 3}\,dx$ $\quad t = \tan\frac{x}{2}$

$\frac{2}{1+t^2}\,dt = dx$

$$= \int \frac{1}{\frac{1-t^2}{1+t^2} - \frac{4t}{1+t^2} + 3} \cdot \frac{2}{1+t^2}\,dt$$

$$= \int \frac{2}{1-t^2-4t+3t^2+3}\,dt$$

$$= \int \frac{2}{4-4t+2t^2}\,dt$$

$$= \int \frac{1}{2-2t+t^2}\,dt$$

$$= \int \frac{1}{1+(t-1)^2}\,dt$$

$$= \tan^{-1}(t-1) + c$$

$$= \tan^{-1}\left(\tan\frac{x}{2} - 1\right) + c.$$

f) $z - 8z + 25 = 0$

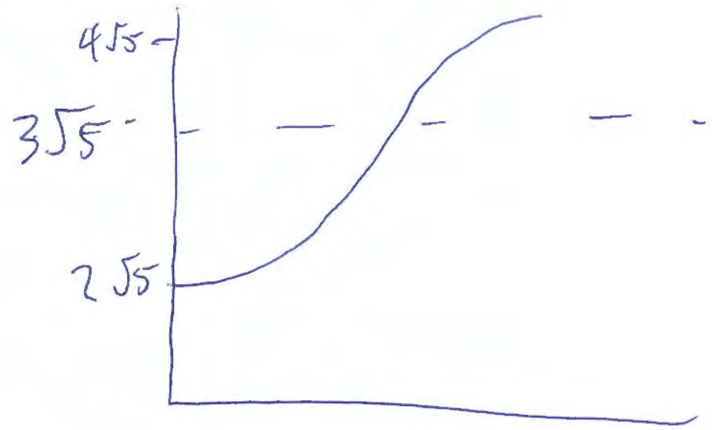
$$z = \frac{8 \pm \sqrt{-36}}{2}$$

$$= \frac{8 \pm 6i}{2}$$

$$= 4 \pm 3i$$

$$= 4 + 3i \text{ or } 4 - 3i$$

9)



period

$$T = 2 \times \frac{\pi}{6}$$

$$= \frac{\pi}{3} = \frac{2\pi}{n}.$$

$$n = \frac{2\pi}{\frac{\pi}{3}}$$

$$n = 6$$

$$A = \frac{4\sqrt{5} - 2\sqrt{5}}{2}$$

Amplitude

$$= \sqrt{5}$$

$$\text{Centre} = \frac{4\sqrt{5} + 2\sqrt{5}}{2}$$

$$= 3\sqrt{5}$$

$$y = -\sqrt{5}\cos 6t + 3\sqrt{5}.$$

Question 12

a) i) $(a-b)^2 \geq 0$

$a^2+b^2-2ab \geq 0$

$a^2+b^2 \geq 2ab$ —— ①

Similarly $a^2+c^2 \geq 2ac$

$b^2+c^2 \geq 2ac$

Adding, we get $2(a^2+b^2+c^2) \geq 2ab+2ac+2abc$

$a^2+b^2+c^2 \geq ab+ac+bc$ ①

2

ii) $(a^2+b^2)(a+b) \geq 2ab(a+b)$ as $(a+b)>0$

$a^3+a^2b+ab^2+b^3 \geq 2ab(a+b)$

$a^3+ab(a+b)+b^3 \geq 2ab(a+b)$

$a^3+b^3 \geq ab(a+b)$ ① ; ①

Similarly $b^3+c^3 \geq bc(b+c)$ ②

$a^3+c^3 \geq ac(a+c)$ ③

①+②+③

$2a^3+2b^3+2c^3 \geq ab(a+b)+bc(b+c)+ac(a+c)$

$2(a^3+b^3+c^3) \geq ab(a+b)+bc(b+c)+ac(a+c)$ ①

2

b) Assume $\sqrt{15}$ rational

that is $\sqrt{15} = \frac{p}{q}$ $p, q \in \mathbb{Z}$ and coprime

$\therefore 15 = \frac{p^2}{q^2}$

$q^2 = \frac{p^2}{15}$

$\therefore p^2$ must be divisible by 15, as $q^2 \in \mathbb{Z}$

$\therefore p$ must be divisible by 15, as $15 = 5 \times 3$, no square factors

(1)

(1)

let $p = 15k$ $k \in \mathbb{Z}$

$\sqrt{15} = \frac{15k}{q}$

$15 = \frac{225k^2}{q^2}$

$q^2 = 15k^2$

(1)

$\therefore q$ is divisible by 15 (no square factors)

$\therefore$ contradiction as p & q are coprime

3

c) $\vec{pq} = (t^3-2t)\underset{\sim}{i} + (4t-2t-3t^2)j + (2t^2-t^3-1)\underset{\sim}{k}$

$2t-3t^2$

$\dot{\vec{pq}} = (3t^2-2)\underset{\sim}{i} + (2\cancel{t}-6t)j + (4t-3t^2)\underset{\sim}{k}$ — 1 differentiating

$\dot{\vec{pq}}(3) = 25\underset{\sim}{i} - 16j - 15\underset{\sim}{k}$ —— 1

$\ddot{\vec{pq}} = (6t)\underset{\sim}{i} \overset{-6}{\cancel{+6}}j + (4-6t)\underset{\sim}{k}$

$\ddot{\vec{pq}}(3) = 18\underset{\sim}{i} - 6j - 14\underset{\sim}{k}$ —— 1

3

d) $\int \ln(x^2-1)\,dx = x\ln(x^2-1) - \int x\left(\frac{2x}{x^2-1}\right)dx$ —— 1

$= x\ln(x^2-1) - \int \frac{2x^2-2}{x^2-1} + \frac{2}{x^2-1}\,dx$ —— 1

$= x\ln(x^2-1) - 2x \cancel{+} \int \frac{1}{x-1} - \frac{1}{x+1}\,dx$ —— 1

$= x\ln(x^2-1) - 2x \cancel{+} \left(\ln|x-1| \cancel{+} \ln|x+1|\right) + c$

$- \ln\left|\frac{x-1}{x+1}\right|$ or $+\ln\left|\frac{x+1}{x-1}\right|$

3

$\int \ln((x-1)(x+1))\,dx = \int \ln(x-1) + \ln(x+1)\,dx$

$= x\ln(x-1) - \int \frac{x}{x-1}\,dx + x\ln(x+1) - \int \frac{x}{x+1}\,dx$

$= x\ln\left|\frac{x-1}{x+1}\right| - \int \frac{x-1}{x-1} + \frac{1}{x-1} + \frac{x+1}{x+1} - \frac{1}{x+1}\,dx$

$= x\ln\left|\frac{x-1}{x+1}\right| - \ln\left|\frac{x+1}{x-1}\right| - 2x + c$

$= (x-1)\ln\left|\frac{x-1}{x+1}\right| - 2x + c$

e) i) As $P(x)$ has all real coefficients, $3-i$ must have a conjugate pair

$\therefore$ $P(x)$ has roots $3-i$, $3+i$, α and β. 1

ii) $3+i+3-i+\alpha+\beta = -\frac{-6}{3}$ 1

$$6+\alpha+\beta = 2$$

$$\alpha+\beta = -4$$

$$\beta = -4-\alpha$$ 1

$$\alpha\beta(3+i)(3-i) = \frac{150}{3}$$

$$10\alpha\beta = 50$$

$$\alpha\beta = 5$$

$$\alpha(-4-\alpha) = 5$$

$$-\alpha^2-4\alpha = 5$$

$$\alpha^2+4\alpha+5 = 0$$

$$\alpha = \frac{-4\pm\sqrt{16-20}}{2}$$

$$= \frac{-4\pm\sqrt{-4}}{2}$$

$$= \frac{-4\pm 2i}{2}$$

$$= -2\pm \cancel{2}i$$

$\therefore$ roots of $P(x)$ are $3+i$, $3-i$, $-2+\cancel{2}i$, $-2-\cancel{2}i$ 2

2

/16

Consider the set $\{3,5\}$

$$3 \times 5 + 1 = 16$$
$$= 4 \times 4$$

which is not prime

$\therefore$ The statement is refuted by a counter example and is untrue.

b) Base case, let $n = 0$. (or ~~zero~~ two depending on ~~defn~~)

$x^0 - y^0 = 0$

which is divisible by $(x+y)$

$n = 2$

$x^2 - y^2 = (x+y)(x-y)$

$\therefore$ true for $n = 2$.

Assume true for $n = k$.

$x^k - y^k = M(x,y)(x+y)$, where $M(x,y)$ is a polynomial

For $n = k+2$

$$\begin{aligned}
LHS &= x^{k+2} - y^{k+2} \\
&= x^2 x^k - y^2 y^k \\
&= x^2 x^k - x^2 y^k + x^2 y^k - y^2 y^k \\
&= x^2(x^k - y^k) + y^k(x^2 - y^2) \\
&= x^2(x+y)M(x,y) + y^k(x+y)(x-y) \quad [\text{By Assu} \\
&= (x+y)[x^2 M(x,y) + y^k]
\end{aligned}$$

$\therefore$ $n=k \Rightarrow n=k+1$

$\therefore$ As $n=0$, $n=2$, $n=k$ and $n=k+2$ are true the statement is true for all $n \geq 0$ ($n \geq 2$) by the principle of Mathematical induction

c) $\underline{u} \cdot \underline{v} = |\underline{u}||\underline{v}| \cos\theta$

$$\cos\theta = \frac{\underline{u} \cdot \underline{v}}{|\underline{u}||\underline{v}|}$$

$$\cos\frac{\pi}{3} = \frac{1\times 2 - 1\times 1 + 3z}{\sqrt{1^2+1^2+z^2} \cdot \sqrt{2^2+1^2+3^2}}$$

$$\frac{1}{2} = \frac{1+3z}{\sqrt{z^2+2}\sqrt{14}}$$

$$\sqrt{z^2+2}\sqrt{14} = 2+6z$$

$$36z^2+24z+4 = 14z^2+28$$

$$22z^2+24z-24=0$$

$$z = \frac{-24 \pm \sqrt{24^2-4\times 22\times -24}}{2\times 22}$$

$$= \frac{-24 \pm 8\sqrt{42}}{44}$$

$$= \frac{-6 \pm 2\sqrt{42}}{11}$$

$$= \frac{-6+2\sqrt{42}}{11} \quad (3z+1>0)$$

d) As z & w are on the unit circle $|z| = |w| = 1$

$$\frac{z - w}{1 - zw} = \frac{(z-w)(1-\overline{zw})}{(1-zw)(1-\overline{zw})}$$

$$= \frac{z - z\bar{z}\bar{w} - w - w\bar{w}\bar{z}}{(1-zw)(1-\overline{zw})}$$

$w\bar{w} = |w|^2 = 1$
$z\bar{z} = |z|^2 = 1$

$$= \frac{z - \bar{w} - w - \bar{z}}{(1-zw)(1-\overline{zw})}$$

$$= \frac{2Re(z) - 2Re(w)}{Real}$$

$(1-zw)(1-\overline{zw})$ is real

$= Real$

e) $F = -0.98\cancel{m}v^2 - g\cancel{m} = \cancel{m}\ddot{x}$

$$\ddot{x} = -\frac{9.8}{100}(100+v^2)$$

$$v\frac{dv}{dx} = -\frac{9.8}{100}(100+v^2)$$

$$\frac{1}{2}\int_u^v \frac{2v}{100+v^2}\,dv = \int_0^y -\frac{9.8}{100}\,dy$$

$$\left[\frac{1}{2}\ln|100+v^2|\right]_u^v = \left[-\frac{9.8}{100}y\right]_0^y$$

$$\frac{1}{2}\left(\ln|100+v^2| - \ln|100+u^2|\right) = -\frac{9.8}{100}y$$

$$\frac{1}{2}\ln\frac{|100+v^2|}{|100+u^2|} = \frac{-9.8}{100}y$$

$$-\frac{250}{49}\ln\frac{|100+v^2|}{|100+u^2|} = y$$

$$y = \frac{250}{49}\ln\frac{|100+u^2|}{|100+v^2|}$$

Question 14:

a) Assume $\exists$ $A \in \mathbb{Z}$ with 2 unique prime factorisations

that is $A = p_1 p_2 p_3 \ldots p_i$ where p_i are prime

and $A = q_1 q_2 q_3 \ldots q_i$ where q_i are prime. (1)

$\therefore p_1 p_2 p_3 \cdots p_i = q_1 q_2 q_3 \cdots q_i$ are equal and A.

as both sides divisible by p_1, and all q_i are prime, then one of q_i must be p_1, say q_1 (1)

$\therefore p_2 p_3 \ldots p_i = q_2 q_3 \ldots q_i$

similarly $p_2 = q_2$ and so on, so each p_i must match q_i and therefore the prime factorisations are the same, a contradiction, so any integer A must have only one unique prime factorisation (1)

/3

b) i) $LHS = \left[F(x)\right]_a^b$

$= F(b) - F(a)$

$RHS = \int_a^b f(a+b-x)\,dx$

$= \left[-F(a+b-x)\right]_a^b$

$= -F(a+b-b) + F(a+b-a)$

$= -F(a) + F(b)$

$= F(b) - F(a)$ as required. (1)

1

ii) $\int_{-1}^{1} \frac{x^2}{1+e^x}\,dx = \int_{-1}^{1} \frac{(1-1-x)^2}{1+e^{-1-x}}\,dx$

$= \int_{-1}^{1} \frac{x^2}{1+e^{-x}}\,dx$ (1)

$= \int_{-1}^{1} \frac{e^x x^2}{1+e^{-x}}\,dx$

$2\int_{-1}^{1} \frac{x^2}{1+e^x}\,dx = \int_{-1}^{1} \frac{e^x x^2}{1+e^x} \quad \frac{x^2}{1+e^x}\,dx.$

$= \int_{-1}^{1} \frac{x^2 \cancel{(e^x+1)}}{\cancel{1+e^x}}\,dx$

$= \int_{-1}^{1} x^2\,dx$

$\int_{-1}^{1} \frac{x^2}{1+e^x}\,dx = 2\int_0^1 x^2\,dx = \left[\frac{x^3}{3}\right]_0^1 = \frac{1}{3}$ // (1) //2

c) $P \begin{pmatrix} 1+\lambda \\ 2-\lambda \\ 1+\lambda \end{pmatrix}$ $Q \begin{pmatrix} 2+2\mu \\ -1+\mu \\ -1+2\mu \end{pmatrix}$ $\overrightarrow{PQ} = \begin{matrix} 2\mu-\lambda+1 \\ \mu+\lambda-3 \\ 2\mu-\lambda-2 \end{matrix}$

$\underset{\sim}{u}_1 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$ $\underset{\sim}{u}_2 = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$

$\overrightarrow{PQ} \cdot \underset{\sim}{u}_1 = 0$ $\overrightarrow{PQ} \cdot \underset{\sim}{u}_2 = 0.$

$2\mu-\lambda+1-\mu-\lambda+3+2\mu-\lambda-2=0.$

$3\mu-3\lambda+2=0$ 1

$4\mu-2\lambda+2+\mu+\lambda-3+4\mu-2\lambda-4=0.$

$9\mu-3\lambda-5=0$

$\mu = \frac{7}{6}$ $\lambda = \frac{11}{6}$ 1

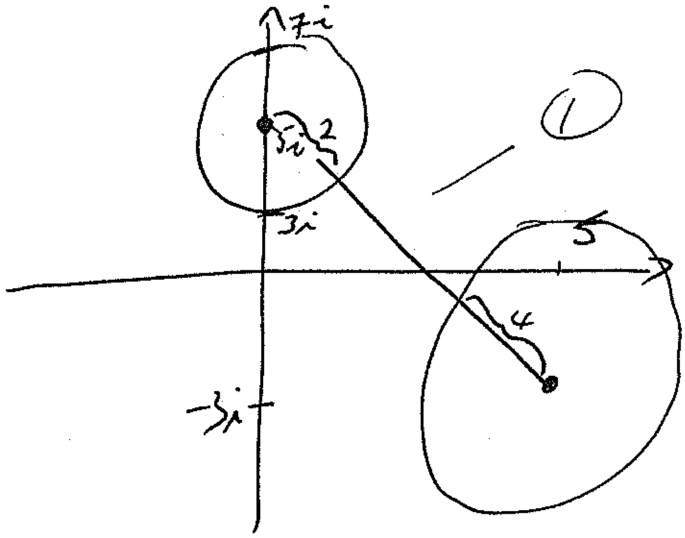
$\overrightarrow{PQ} = \begin{pmatrix} 3/2 \\ 0 \\ -3/2 \end{pmatrix}$ distance $\Rightarrow \sqrt{\frac{9}{4}+\frac{9}{4}}$

$= \sqrt{\frac{18}{4}}$

$= \frac{3\sqrt{2}}{2}$ 1

3

d)



(1)

$$|C_1 - C_2| = \sqrt{8^2 + 5^2}$$
$$= \sqrt{89} \quad (1)$$

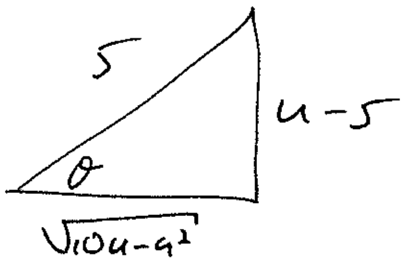
$$\text{Min} = \sqrt{89} - 6$$
$$\text{Max} = \sqrt{89} + 6. \quad (1)$$

//3

e) $\int e^x \sqrt{10e^x - e^{2x}}\, dx$

$e^x = u \qquad e^x\,dx = du \quad (1)$

$$= \int \sqrt{10u - u^2}\, du$$
$$= \int \sqrt{25 - (u^2 - 10u + 25)}\, du$$
$$= \int \sqrt{25 - (u-5)^2}\, du \quad (1)$$



$u - 5 = 5\sin\theta$

$du = 5\cos\theta\, d\theta$

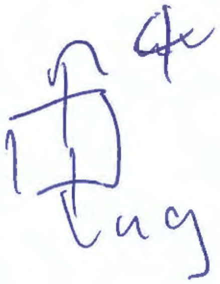
$$\int 5\cos\theta \cdot 5\cos\theta\, d\theta$$
$$25\int \cos^2\theta\, d\theta$$
$$= \frac{25}{2}\int 1 + \cos 2\theta\, d\theta$$
$$= \frac{25}{2}\left(\theta + \frac{\sin 2\theta}{2}\right) + c \quad (1)$$

$$= \frac{25}{2}\left(\sin^{-1}\frac{u-5}{5} + \frac{u-5}{5}\cdot\frac{\sqrt{10u-u^2}}{5}\right) + c$$

$$= \frac{25}{2}\left(\sin^{-1}\left(\frac{e^x-5}{5}\right) + \frac{e^x-5}{25}\left(\sqrt{10e^x - e^{2x}}\right)\right) + c \quad (1)$$

//4

5 a i)



$$F = mg - 4v.$$

$$\cancel{2a} = 2g - 4v = 2a.$$

$$a = g - 2v.$$

ii) $$\frac{dv}{dt} = g - 2v.$$

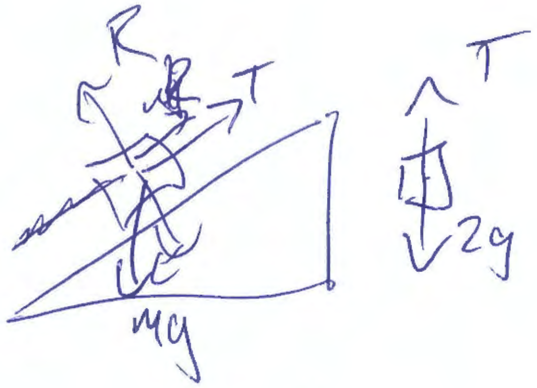
$$\int_0^v \frac{1}{g-2v}\,dv = \int_0^t dt$$

$$\left[-\tfrac{1}{2}\ln|g-2v|\right]_0^v = [t]_0^t$$

$$-\tfrac{1}{2}\left(\ln|g-2v| - \ln|g|\right) = t$$

$$t = -\tfrac{1}{2}\ln\frac{|g-2v|}{|g|}$$

b i



Tension needed to be present.

Friction had to go up the plane (consider case with no friction)

Forces Along Plane.

i $F_5 = 5g\sin 30° - 0.3(5g\cos 30°) - T = 5a$

$F_2 = T - 2g = 2a$

$F_2 + F_5$

$5g\sin 30 - 0.3(5g\cos 30° - 2g = 7a$

$\left(\frac{1}{2} - \frac{3\sqrt{3}}{4}\right)gN = 7a$

$5a = \frac{5}{7}\left(\frac{1}{2} - \frac{3\sqrt{3}}{4}\right)gN.$

c i) $I_n = \int_0^a (a-x)^n \cos x \, dx$

$= \left[(a-x)^n \sin x\right]_0^a + n\int_0^a (a-x)^{n-1} \sin x \, dx$

$= 0 + n\left(\left[-(a-x)^{n-1}\cos x\right]_0^a - (n-1)\int_0^a (a-x)^{n-2} \cos x \, dx\right)$

$= n\left(-(-a^{n-1}) - n-1 I_{n-2}\right)$

$= na^{n-1} - n(n-1) I_{n-2}$

ii) $I_0 = \int_0^{\frac{\pi}{2}} \cos x \, dx$

$= \left[\sin x\right]_0^{\frac{\pi}{2}}$

$= 1$

$I_2 = 2\left(\frac{\pi}{2}\right)^{2-1} - 2(2-1) I_0$

$= \pi - 2$

Question 16

a) i) If $\omega_1 \dots \omega_n$ are complex roots of the equation $z^n - 1 = 0$, then $\omega_1 + \omega_2 + \dots + \omega_n = 0$ (sum of the roots and the coefficient of z^{n-1} is zero).

1

1

ii) $|PA_i| = |z - \omega_i|$

$$PA_i^2 = (z-\omega_i)^2$$

$$= (z-\omega_i)\overline{(z-\omega_i)}, \text{ since } |z|^2 = z \cdot \bar{z}$$

$$= (z-\omega_i)(\bar{z} - \bar{\omega}_i)$$

1

1

iii) $$\sum_{i=1}^{n} PA_i^2 = \sum_{i=1}^{n} (z-\omega_i)(\bar{z}-\bar{\omega}_i)$$

1

$$= \sum_{i=1}^{n} (z\bar{z} - z\bar{\omega}_i - \omega_i \bar{z} + \omega_i\bar{\omega}_i)$$

$$= \sum_{i=1}^{n} (2 - z\bar{\omega}_i - \omega_i\bar{z})$$

1

since $z\bar{z} = |z|^2 = 1$; $\omega\bar{\omega} = |\omega|^2 = 1$

$$= 2n - z\sum_{i=1}^{n} \bar{\omega}_i - \bar{z}\sum_{i=1}^{n} \omega_i$$

$$= 2n - z \cdot 0 - \bar{z} \cdot 0 \quad \text{by (i)}$$

$$= 2n$$

1

3

b) $f(x) = 1+x^2, \quad f'(x) = 2x$

$$x_{n+1} = x_n - \frac{1+x_n^2}{2x_n} \quad \text{— (1)}$$

i) $(1-x_n)^2 \geq 0$

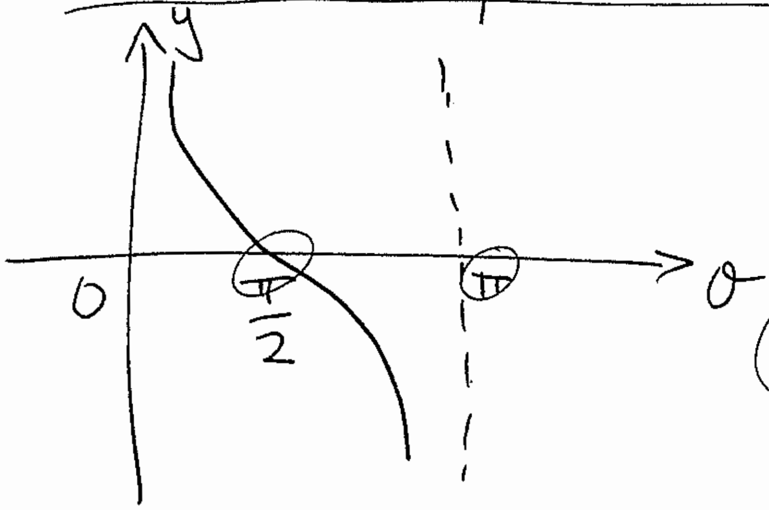
$$\Rightarrow 1 + x_n^2 \geq 2x_n$$

$$\frac{1+x_n^2}{2x_n} \geq 1$$

$$\therefore |x_{n+1} - x_n| = \left|\frac{1+x_n^2}{2x_n}\right| \text{ from (1)}$$

1

ii)



2

iii) cot θ takes all real values exactly once in the interval $0 < \theta < \pi$, hence there exists a real number θ_n for all real x_n

1

iv) $\tan 2\theta = \dfrac{2\tan\theta}{1-\tan^2\theta}$

$\therefore \cot 2\theta = \dfrac{1-\tan^2\theta}{2\tan\theta}$.

$= \dfrac{\cot^2\theta - 1}{2\cot\theta}$ divide by $\tan^2\theta$.

(1)

hence $\cot\theta_{n+1} = x_{n+1}$

$= x_n - \dfrac{1+x_n^2}{2x_n}$

$= \cot\theta_n - \dfrac{1+\cot^2\theta_n}{2\cot\theta_n}$

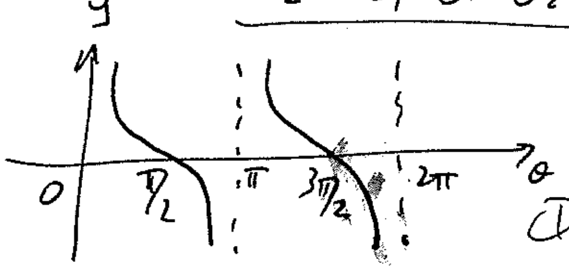
$= \dfrac{\cot^2\theta_n - 1}{2\cot\theta_n}$

$= \cot 2\theta_n$ (1)

/2

v) from (iv): $\cot\theta_2 = \cot 2\theta_1$

$\Rightarrow \theta_2 = 2\theta_1$ or $\theta_2 = 2\theta_1 - \pi$ as $0 < \theta < \pi$ (1).



(1)

ie $\theta_2 = 2\theta_1 - m_1\pi$ $[m_1 = 0, 1]$

$\therefore \theta_3 = 2\theta_2$

$= 4\theta_1 - m_2\pi$ $[m_2 = 0, 1, 2]$ (1)

$\therefore \theta_{n+1} = 2^n\theta_1 - m\pi$ $[m = 0, 1, \ldots n]$

If $x_1 = x_{n+1}$, then $\theta_1 = \theta_{n+1}$

from (ii) $\theta_1 = 2^n\theta_1 - m\pi$.

$\theta_1 = \dfrac{m\pi}{2^n - 1}$ (1) hence $x_1 = \cot\theta_1 = \cot\dfrac{m\pi}{2^n-1}$ $\left[\begin{array}{l} m, n \in \mathbb{Z} \\ n \geq 1, 0 \leq m \leq \ldots \end{array}\right.$

/4

Observe

$m=1, n=2 \qquad \cot\theta_1 = \cot \pi/3 = \frac{1}{\sqrt{3}}$

$x_1 = \frac{1}{\sqrt{3}} \qquad x_{n+1} = x_n - \frac{1+x_n^2}{2x_n}$

$x_2 = \frac{1}{\sqrt{3}} - \frac{1+\frac{1}{3}}{2\left(\frac{1}{\sqrt{3}}\right)}$

$= \frac{1}{\sqrt{3}} - \frac{4/3}{2/\sqrt{3}} = \frac{1}{\sqrt{3}} - \frac{4}{3} \times \frac{\sqrt{3}}{2}$

$= \frac{1}{\sqrt{3}} - \frac{2}{\sqrt{3}} = -\frac{1}{\sqrt{3}}$

$x_3 = -\frac{1}{\sqrt{3}} - \frac{1+\frac{1}{3}}{2\left(-\frac{1}{\sqrt{3}}\right)}$

$= -\frac{1}{\sqrt{3}} + \frac{4/3}{2/\sqrt{3}}$

$= -\frac{1}{\sqrt{3}} + \frac{2}{\sqrt{3}} = \frac{1}{\sqrt{3}}$

for $n=2$, $x_1 = x_3$